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Ω-INVERSE LIMIT STABILITY THEOREM

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Dedicated to the memory of my father

Abstract. We prove that if an endomorphism f satisfies weak Axiom A and the no-cycles condition then f is Ω-inverse limit stable. This result is a generalization of Smale's Ω-stability theorem from diffeomorphisms to endomorphisms.

1. Introduction

In the theory of dynamical systems, research of orbit structure is one of main subjects. The central role of this study is played by the topological conjugacy. Two continuous maps $f_i : X_i \hookleftarrow$ of topological spaces X_i, i=1,2, are *topologically conjugate* if there exists a homeomorphism $h : X_1 \to X_2$ such that $f_2 h = h f_1$. If h is only continuous surjective, then h is called a *semiconjugacy*. The image of an f_1-orbit by a semiconjugacy is an f_2-orbit, while a topological conjugacy sends f_1-orbits to f_2-orbits and preserves their topological properties. Let M be a smooth compact connected boundaryless manifold and $\mathrm{Diff}^r(M)$ (resp. $\mathrm{End}^r(M)$) the space of C^r diffeomorphisms (resp. endomorphisms) of M endowed with the C^r topology, $r \geq 1$. We say that $f \in \mathrm{Diff}^r(M)$ ($\mathrm{End}^r(M)$) is C^r *structurally stable* if every g near f is topologically conjugate to f. Closely related to structural stability is Ω-stability. We say that f is Ω-*stable* if every g near f, $g|\Omega(g)$ is topologically conjugate to $f|\Omega(f)$. Here $\Omega(f)$ denotes the set of nonwandering points (i.e. the points $x \in M$ such that for every neighborhood V of x there exists $n > 0$ satisfying $f^n(V) \cap V \neq \phi$).

From now on we restrict our attention to Ω-stability for diffeomorphisms and endomorphisms. We shall outline the development of Ω-stability for discrete systems without giving precise definitions. Smale [13] proved that a diffeomorphism satisfying Axiom A and the no-cycles condition is Ω-stable. Przytycki [9] obtained the similar result for endomorphisms. Przytycki's conditions for Ω-stability require that the nonwandering set contains no singularities. Recall the definition of singularity: $x \in M$ is a *singularity* of $f \in \mathrm{End}^r(M)$ if $T_x f$ is not injective. However, there is an example of Ω-stable endomorphisms whose nonwandering sets persistently contain the singularities [7].

Recently we obtained that if an endomorphism f has a neighborhood $\mathcal{U}$ such that every g in $\mathcal{U}$ satisfies weak Axiom A then f is Ω-stable [5]. The above sufficient condition allows the existence of singularities in the nonwandering set. In fact

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our condition is weaker than Przytycki's conditions: our condition plus $\Omega(f) \cap S(f) = \phi$ is equivalent to Axiom A plus the no-cycles condition (that is, Przytycki's conditions). Here $S(f)$ denotes the set of singularities of $f \in \mathrm{End}^r(M)$.

In the case of diffeomorphisms the following are equivalent:
(a) $f \in \mathrm{Diff}^r(M)$ satisfies Axiom A and the no-cycles condition;
(b) $f \in \mathrm{Diff}^r(M)$ has a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^r(M)$ such that every g in $\mathcal{U}$ satisfies Axiom A.

Our sufficient condition for Ω-stability of endomorphisms corresponds to (b) above of diffeomorphisms. Hence our result is a natural generalization of Smale's Ω-stability theorem from diffeomorphisms to endomorphisms. In the case of endomorphisms it is natural that we pose whether the following are equivalent:
(c) $f \in \mathrm{End}^r(M)$ satisfies weak Axiom A and the no-cycles condition;
(d) $f \in \mathrm{End}^r(M)$ has a neighborhood $\mathcal{U}$ of f in $\mathrm{End}^r(M)$ such that every g in $\mathcal{U}$ satisfies weak Axiom A.

It is easy to see that (d) implies (c). However it is not known whether (c) implies (d). In this paper we show that (c) implies a sort of Ω-stability, i.e. "inverse limit stability" on the nonwandering set.

Theorem. *If $f \in \mathrm{End}^r(M)$ satisfies weak Axiom A and the no-cycles condition then f is C^r Ω-inverse limit stable.*

Inverse limit stability does not preserve the topological dynamics. However inverse limit stability gives the one-to-one correspondence between all bi-infinite orbits for any two endomorphisms near original one. In fact, inverse limit stability is a generalization of structural stability. Because the two concepts coincide for diffeomorphisms.

The concept of inverse limit stability was introduced by Mañé and Pugh [7] and Przytycki [8] in connection with nonsingular endomorphisms of compact manifolds. Quandt [10] proved an extension of their results for Anosov maps by allowing singularities and Banach manifolds. For more detailed motivation and explanation in this direction see Quandt [11].

The contents of this paper are as follows: In §2 we give some definitions and theorems to develop endomorphism theory. In Subsection 2.1 we define prehyperbolic sets and weak Axiom A. Moreover we relate these notions with hyperbolic sets and Axiom A for diffeomorphisms. In Subsection 2.2 we give the definition of inverse limit stability and P-hyperbolic structure. In Subsection 2.3 we develop the theory of stable and unstable sets for prehyperbolic sets. Moreover we show that prehyperbolic sets with a local product structure have the similar properties to those for hyperbolic sets with a local product structure, e.g., Shadowing Lemma, local maximality. At last we introduce the no-cycles condition for a weak Axiom A endomorphism and prove the filtration lemma. In §3 we prove the Theorem. In §4 we state some remarks on the relation between infinitesimal stability and Ω-inverse limit stability.

2. Preliminaries

In this section we give some definitions and theorems to develop endomorphism theory.

2.1. **Prehyperbolic sets.**

Definition 1 ([6])**.** Let $f \in \mathrm{End}^r(M)$ and let Λ be a compact subset of M with $f(\Lambda) = \Lambda$. We say that Λ is a *prehyperbolic set* for f if there exist a continuous splitting $TM|\Lambda = E^s \oplus E^u$, and a Riemannian norm $|\ \ |$ on TM and constants $K > 0$, $0 < \lambda < 1$ satisfying:

(a) $(Tf)E^s \subset E^s$, $(Tf)E^u = E^u$;

(b) $|(Tf)^n v| \le K\lambda^n |v|$ for $x \in \Lambda$, $v \in E^s_x$, $n > 0$,

$$|(Tf)^n v| \ge K\lambda^{-n}|v| \text{ for } x \in \Lambda, v \in E^u_x, n > 0;$$

(c) if $x_1 \ne x_2 \in \Lambda$ and $f(x_1) = f(x_2) = y$, $E^s_y = \{0\}$.

Definition 2. We say that $f \in \mathrm{End}^r(M)$ satisfies *weak Axiom A* if

(a) the periodic points of f are dense in $\Omega(f)$;

(b) $\Omega(f)$ is prehyperbolic for f.

Condition (a) guarantees f-invariance of $\Omega(f)$, i.e., $f(\Omega(f)) = \Omega(f)$. Remark that in general $f(\Omega(f)) \subset \Omega(f)$.

Remark. If a weak Axiom A endomorphism f satisfies $\Omega(f) \cap S(f) = \phi$ then we say that f satisfies Axiom A [6].

If a weak Axiom A endomorphism f is a diffeomorphism then f satisfies Smale's Axiom A [13]. That is, weak Axiom A for endomorphisms is a natural generalization of Axiom A for diffeomorphisms.

We say that a periodic point x of $f \in \mathrm{End}^r(M)$ with period p is *prehyperbolic* if $Tf^p : T_xM \to T_xM$ has no eigenvalues of absolute value 1. Then let $E^u(x)$ be the subspace of T_xM associated to the eigenvalues of $Tf^p : T_xM \hookleftarrow$ that have absolute value > 1. We call $\dim M - \dim E^u(x)$ the stable index of x for f. Similarly if Λ is a prehyperbolic set with $\dim E^s_x = j$ for all $x \in \Lambda$ then we call j the *stable index* of Λ for f. If a prehyperbolic set Λ has a positive stable index, then f restricted to Λ is one to one by condition (c) of Definition 1. If it is further assumed that there are no singularities in Λ, then Λ is hyperbolic in the sense of diffeomorphisms. Let $\mathcal{P}F^r(M)$ be the interior of the set of all C^r endomorphisms of M such that every periodic point is prehyperbolic.

In the proof of the theorem we shall use the following results for prehyperbolic sets.

Theorem 2.1. *Let Λ be a prehyperbolic set for a C^r endomorphism f of M, $r \ge 1$. There are numbers $\alpha > 0$, $K > 0$, $k > 0$, a neighborhood U of Λ in M and a neighborhood V of f in $\mathrm{End}^r(M)$ with the following properties:*

For any topological space X, any homeomorphism h of X, and any continuous map $i : X \to U$, if g belongs to V and $d(ih, gi) \le \alpha$, then there is a unique continuous map $j : X \to M$ such that $jh = gj$ and $d(i, j) \le k$. In fact, we have the *strong estimate that* $d(i,j) \leq Kd(ih, gi)$*. Moreover, for fixed* i *and* h*,* j *depends* C^0 *continuously on* g*. Here* $d(i,j) = \sup\{\rho(i(x), j(x)) | x \in X\}$*, where* ρ *is a metric on* M*.*

Theorem 2.2. *Let* Λ *be a prehyperbolic set for* $f \in \mathrm{End}^r(M)$*,* $r \geq 1$*. There is a neighborhood* V *of* f *in* $\mathrm{End}^r(M)$ *and a continuous function* $\Phi : V \to C^0(\Lambda, M)$ *such that*:

(1) $\Phi(f)$ *is the inclusion of* Λ *in* M*;*
(2) $\Phi(g)(\Lambda)$ *is a P-hyperbolic g-invariant set for any* g *in* V*;*
(3) $\Phi(g)$ *is a semiconjugacy of* Λ *onto* $\Phi(g)(\Lambda)$*, that is,*

$$g \circ \Phi(g) = \Phi(g) \circ f \quad on \quad \Lambda\ ;$$

(4) *There is a constant* $K > 0$ *such that*

$$d_{C^0}(\Phi(g), inc_\Lambda) \leq K d_{C^0}(f, g).$$

Theorem 2.1 is proved following the case of diffeomorphisms in Chapter 7 of [12]. (1), (3) and (4) of Theorem 2.2 are proved by the similar argument in [12] with Theorem 2.1. (2) of Theorem 2.2 will be proved in §3.

2.2. **Inverse limit stability.** Let M be a compact connected smooth manifold without boundary. With M^Z we denote the class of all maps from Z to M, and choose

$$\tilde{d}(v, w) = \sum_{i=-\infty}^{\infty} 2^{-|i|} d(v(i), w(i)) \quad \text{for } v, w \in M^Z$$

as a metric on M^Z, where d is a metric on M induced by a Riemannian metric. This metric induces the product topology on M^Z.

Any endomorphism $f : M \to M$ induces a map $\tilde{f} : M^Z \to M^Z$ through

$$(\tilde{f}(v))(i) = f(v(i)) \quad \text{for} \quad v \in M^Z,$$

which we call the lift of f to M^Z.

Given a subset Λ of M with $f(\Lambda) \subset \Lambda$, we define

$$\widetilde{\Lambda}(f) = \{v \in \Lambda^Z | f(v(i)) = v(i+1) \ \text{ for all } i \in Z\},$$

that is the set of all bi-infinite orbits of f contained in Λ, and

$$\widetilde{\mathcal{O}}(f) = \{v \in M^Z | f(v(i)) = v(i+1) \ \text{ for all } i \in Z\},$$

that is the set of all bi-infinite orbits of f.

Moreover, we set $A(f) = \bigcap_{n\geq 0} f^n(M)$. We will say that a subset Λ of M is ω-invariant for f if $f(\Lambda) \subseteq \Lambda$.

Definition 3 ([10])**.** We say that two C^r endomorphisms f and g of M are *inverse limit conjugate* if there exists a homeomorphism $H : \widetilde{\mathcal{O}}(f) \to \widetilde{\mathcal{O}}(g)$ such that $H\tilde{f} = \tilde{g}H$ on $\widetilde{\mathcal{O}}(f)$. A C^r endomorphism f of M is called C^r *inverse limit stable* if there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{End}^r(M)$ such that for every g in $\mathcal{U}$, f and g are inverse limit conjugate. Similarly, a C^r endomorphism f of M is called C^r Ω-*inverse limit stable* if there exists a C^r neighborhood $\mathcal{U}$ of f in $\mathrm{End}^r(M)$ such that for every g in $\mathcal{U}$ there is a homeomorphism $H : \widetilde{\Omega(f)} \to \widetilde{\Omega(g)}$ with $H\tilde{f} = \tilde{g}H$ on $\widetilde{\Omega(f)}$. Here $\widetilde{\Omega(f)}$ is the set of all bi-infinite orbits of f with values in $\Omega(f)$.

The concept of (Ω-) inverse limit stability for endomorphisms is a natural extension of the concept of structural(Ω-) stability for diffeomorphisms, i.e., for diffeomorphisms the two concepts coincide.

Definition 4 ([10])**.** We say that a C^r endomorphism f of M has a *P-hyperbolic structure* for an ω-invariant set Λ if there are constants $C > 0$, $0 < \mu < 1$, a Riemannian norm $|\ |$ on TM such that for every $v \in \widetilde{\Lambda}(f)$, there exist a splitting of $\bigcup_{i\in Z} T_{v(i)}M$ into a direct sum $E^s \oplus E^u$ satisfying:

(1) $E^s = \bigcup_{i=-\infty}^{\infty} E^s_{v(i)}$ and $E^u = \bigcup_{i=-\infty}^{\infty} E^u_{v(i)}$

$$(T_{v(i)}f)E^s_{v(i)} \subset E^s_{v(i+1)} \quad \text{and} \quad (T_{v(i)}f)E^u_{v(i)} = E^u_{v(i+1)}$$

for all $i \in Z$;

(2) $|(T_{v(i)}f)^n w| \le C\mu^n |w|$ for all $i \in Z$, $n \in Z^+$ and $w \in E^s_{v(i)}$

$|(T_{v(i)}f)^n w| \ge C^{-1}\mu^{-n}|w|$ for all $i \in Z$, $n \in Z^+$ and $w \in E^u_{v(i)}$.

Remark that the definition of P-hyperbolic structure dose not imply a continuous splitting of the whole restricted tangent bundle of M over Λ. If $f|\Lambda$ is a homeomorphism of Λ onto itself then P-hyperbolicity of Λ implies prehyperbolicity of Λ.

P-hyperbolic sets have a kind of inverse limit stability as follows:

Theorem 2.3 ([10])**.** *Let $I(f)$ be an ω-invariant set for $f \in \mathrm{End}^r(M)$. Suppose that f has a P-hyperbolic structure for $I(f)$ and $\bigcap_{n\ge 0} f^n(I(f))$ is compact in M. Then there exists a C^r neighborhood $\mathcal{U}$ of f in $\mathrm{End}^r(M)$ such that for every g in $\mathcal{U}$ there is an ω-invariant $J(g)$ for g such that there exists a homeomorphism H : $\widetilde{I(f)} \to \widetilde{J(g)}$ with $H\tilde{f} = \tilde{g}H$ on $\widetilde{I(f)}$. By choosing the neighborhood $\mathcal{U}$ sufficiently small we will have H arbitrarily close to the identity. Subject to this restriction, the conjugacy is unique.*

2.3. **Stable and unstable sets for prehyperbolic sets.** Let Λ be a prehyperbolic set for $f \in \mathrm{End}^r(M)$ with the stable index $j > 0$. Let d be the topological metric on M induced by some Riemannian metric. For $x \in \Lambda$, $\varepsilon > 0$, the *local stable and unstable sets* are defined by

$W^s_\varepsilon(x) = \{y \in M | d(f^n(x), f^n(y)) \le \varepsilon$ for $n \ge 0$ and $d(f^n(x), f^n(y)) \to 0$ as $n \to +\infty\}$,

$W^u_\varepsilon(x) = \{y \in M|$ there exists a sequence $\{y_{-n}|\ n \ge 0\}$ such that $y_0 = y$, $f(y_{-n}) = y_{-n+1}$ and $d(y_{-n}, x_{-n}) \le \varepsilon$ for $n \ge 0$ and $d(y_{-n}, x_{-n}) \to 0$ as $n \to +\infty$, where $\{x_{-n}|n \ge 0\} \subset \Lambda$ is a unique negative orbit of x contained in $\Lambda\}$.

Note in the definition of unstable sets the existence and uniqueness of the above negative orbit of x is guaranteed by the stable index of $\Lambda > 0$.

From the well-known results of [2], the following properties are obtained:
(a) $W^\sigma_\varepsilon(x)$ is tangent to E^σ_x for $\sigma = s, u$;
(b) $W^s_\varepsilon(x)$ is a j-dimensional C^r disk;
(c) $W^u_\varepsilon(x)$ is a $(\dim M - j)$-dimensional C^r disk.

The stable and unstable sets also satisfy the following:

(1) The embedding of $W_\varepsilon^\sigma(x)$ varies continuously with x in Λ for $\sigma = s, u$.

(2) For an adapted Riemannian metric which induces d,

$d(f^n(x), f^n(y)) \le \lambda^n d(x,y)$ for $y \in W_\varepsilon^s(x)$, $n \ge 0$;

$d(y_{-n}, x_{-n}) \le \lambda^n d(x,y)$ for $y \in W_\varepsilon^u(x)$, $n \ge 0$,

where $0 < \lambda < 1$ is such that $||Tf|E^s|| < \lambda$ and $||(Tf|E^u)^{-1}|| < \lambda$.

(3) $W_\varepsilon^s(x) = \{y \in M | d(f^n(x), f^n(y)) \le \varepsilon$ for all $n \ge 0\}$.

$W_\varepsilon^u(x) = \{y \in M|$ there is a unique negative orbit $\{y_{-n}\}$ of y such that $y_0 = y, d(y_{-n}, x_{-n}) \le \varepsilon$ for $n \ge 0\}$.

(4) $\{W_\varepsilon^\sigma(x)|x \in \Lambda\}$ is locally f-invariant for $\sigma = s, u$; for every $x \in \Lambda$

$$f(W_\varepsilon^s(x)) \subset W_\varepsilon^s(f(x)), \quad W_\varepsilon^u(f(x)) \subset f(W_\varepsilon^u(x)).$$

(5) $\{W_\varepsilon^\sigma(x)|x \in \Lambda\}$ is *self-coherent* for $\sigma = s, u$, i.e. the interior of each pair of its discs meet in a relatively open subset of each.

Local stable and unstable sets for prehyperbolic sets of endomorphisms have nice properties and structures similar to those for local stable and unstable manifolds for hyperbolic sets of diffeomorphisms.

Let us define the *stable and unstable sets* of x in Λ as

$W^s(x) = \bigcup_{n \ge 0} V_n(x)$, where $V_n(x)$ is a connected component of the preimage of $W_\varepsilon^s(f^n(x))$ under f^n containing $W_\varepsilon^s(x)$;

$W^u(x) = \bigcup_{n \ge 0} f^n(W_\varepsilon^u(x_{-n}))$ where $\{x_{-n}|n \ge 0\}$ is a unique negative orbit of x contained in Λ.

Moreover we define more general objects,

$\widetilde{W}^s(x) = \{y \in M|\ d(f^n(x), f^n(y)) \to 0$ as $n \to +\infty\}$

$\widetilde{W}^u(x) = \{y \in M|$ there exists a sequence $\{y_{-n}|n \ge 0\}$ such that $y_0 = y$, $f(y_n) = y_{-n+1}$ and $d(y_{-n}, x_{-n}) \to 0$ as $n \to +\infty$, where $\{x_{-n}|n \ge 0\}$ is the above negative orbit of x contained in $\Lambda\}$.

We call $\widetilde{W}^s(x)$ ($\widetilde{W}^u(x)$) the *generalized stable* (*unstable*) *set* of x in Λ. $\widetilde{W}^s(\Lambda) = \bigcup_{x \in \Lambda} \widetilde{W}^s(x)$ is called the *generalized stable set* of Λ. Similarly for $\widetilde{W}^u(\Lambda) = \bigcup_{x \in \Lambda} \widetilde{W}^u(x)$.

Let $\widetilde{\widetilde{W^s}}(\Lambda) = \{y \in M| d(f^n(y), \Lambda) \to 0$ as $n \to +\infty\}$ and $\widetilde{\widetilde{W^u}}(\Lambda) = \{y \in M|$ there exists an infinitely negative orbit $\{y_{-n}\}$ such that $d(y_{-n}, \Lambda) \to 0$ as $n \to +\infty\}$.

We call $\widetilde{\widetilde{W^s}}(\Lambda)$ ($\widetilde{\widetilde{W^u}}(\Lambda)$) the weakly stable (unstable) set of Λ.

It is obvious that in general for $\sigma = s, u$,

(a) $W^\sigma(x) \subset \widetilde{W}^\sigma(x)$ for every x in Λ;

(b) $W^\sigma(\Lambda) \subset \widetilde{W^\sigma}(\Lambda) \subset \widetilde{\widetilde{W^\sigma}}(\Lambda)$.

Definition 5 ([1, 12])**.** Let X be a metric space with a metric d and f a continuous map of X to itself. We say that f is *expansive* on a subset Y of X if there is a constant $\varepsilon > 0$ such that for any pair of bi-infinite orbits $\underline{x}$ and $\underline{y}$ with $\underline{x} \subset X$, $\underline{y} \subset Y$: if $d(x_n, y_n) < \varepsilon$ for all $n \in Z$ then $\underline{x} = \underline{y}$. For $\alpha > 0$ an *α-pseudo-orbit* for f is a sequence $\{x_i | -\infty \le a < i < b \le +\infty\}$ such that $d(f(x_i), x_{i+1}) < \alpha$ for $a < i < b - 2$.

Definition 6 ([1, 12])**.** Let X be a metric space with a metric d, and f a homeomorphism of X to itself. Let $\underline{x} = \{x_i | a < i < b\}$ be an α-pseudo-orbit of f. One say that a point x' *β-shadows* $\underline{x}$ if $d(f^i(x'), x_i) < \beta$ for $a < i < b$.

By the similar arguments to those for hyperbolic sets of diffeomorphisms [12], we have the following properties of prehyperbolic sets:

Let Λ be a prehyperbolic set of $f \in \mathrm{End}^r(M)$, $r \ge 1$.

Property 1. For every small positive η there is a positive δ such that if x, y in Λ satisfy $d(x, y) < \delta$ then $W^s_\eta(x) \cap W^u_\eta(y) = \{p\}$, p is a point of transverse intersection of $W^s_\eta(x)$ and $W^u_\eta(y)$. We denote this point p by $[x, y]_{\eta,\delta}$.

Property 2. f is expansive on Λ.

Property 3. If Λ has a dense subset of periodic points of f, Λ has a *local product structure*, i.e. for small ε and δ,

$$[x, y]_{\varepsilon,\delta} \text{ belongs to } \Lambda \text{ whenever } d(x, y) < \delta.$$

We shall prove that prehyperbolic sets with a local product structure have good properties similar to those for hyperbolic sets with a local product structure. A weak Axiom A endomorphism has a decomposition of the nonwandering set into prehyperbolic sets with a local product structure. Hence the following propositions are useful for weak Axiom A endomorphisms.

The following is the Shadowing Lemma. This is proved by the similar arguments for diffeomorphisms. However endomorphisms case needs more estimates by the self-coherence of unstable disk family.

Proposition 2.4 (Shadowing Lemma)**.** *Let Λ be a prehyperbolic set for $f \in \mathrm{End}^r(M)$, $r \ge 1$. Suppose that Λ has a local product structure and the stable index $i > 0$. For every small $\beta > 0$ there is an $\alpha > 0$ such that every α-pseudo-orbit $\underline{x}$ in Λ is β-shadowed by a point y of Λ.*

Proof. First remark that $h = f|\Lambda$ is a homeomorphism of Λ to itself. Moreover h and h^{-1} are uniformly continuous. Suppose that M has an adapted metric. Choose ε as in the stable and unstable sets for Λ, and let $0 < \lambda < 1$ be the constant of prehyperbolicity of Λ. Now choose a positive $\varepsilon_1 < (1 - \lambda) \min\{\varepsilon, \beta\}$.

Let $\eta = \varepsilon_1/(1-\lambda)$ and let δ be a positive constant less than $\beta - \eta$ for which $[\cdot,\cdot]_{\varepsilon_1,\delta} : U_\delta(\Delta_\Lambda) \to \Lambda$ defines a local product structure. Since $[\cdot,\cdot]$ is continuous, as is $W^s_\varepsilon(\cdot)$, it makes sense to define α by requiring that whenever z and w in Λ are α-close

$$[z, W^s_{\lambda\delta}(w) \cap \Lambda] \subset W^s_\delta(z).$$

Take first the α-pseudo-orbit $\underline{x}$ which has the form $\{x_0, \cdots, x_n\}$. Set $y_0 = x_0$ and define y_k inductively by $y_k = [x_k, f(y_{k-1})]$ for $1 \le k \le n$.

The above definition needs that y_k belongs to $W^s_\delta(x_k) \cap \Lambda$. For $k = 1$ it is trivial. Suppose by induction that y_{k-1} belongs to $W^s_\delta(x_{k-1}) \cap \Lambda$. Then $f(y_{k-1})$ belongs to $W^s_{\lambda\delta}(f(x_{k-1})) \cap \Lambda$ and hence y_k belongs to $W^s_\delta(x_k)$, so our definition is valid. Then we can take an orbit $\{\tilde{y}_0, \cdots, \tilde{y}_{n-1}, y_n\}$ such that $\tilde{y}_{n-j}$ belongs to $W^u_{\theta_j}(y_{n-j})$ for all $1 \le j \le n$ where $\theta_j = \sum_{i=1}^{j} \lambda^j \varepsilon_1 \le \eta = \varepsilon_1/(1-\lambda)$. It is easy to see that $\tilde{y}_0$ β-shadows $\underline{x}$ [12]. However we do not know whether $\tilde{y}_0$ belongs to Λ. Let ζ be a positive constant such that $[\cdot,\cdot]_{\varepsilon_1,\zeta}$ is well-defined. Since h^{-1} is uniformly continuous, there is $0 < r < \varepsilon_1$ such that if x, y in Λ satisfy $d(x,y) < r$ then $d(h^{-1}(x), h^{-1}(y)) < \zeta$. If necessary, we retake α such that $d(f(y_{k-1}), y_k) < r$ for all $1 \le k \le n$.

Remark that we can take α independent of length n because Λ is compact and $[\cdot,\cdot]_{\varepsilon_1,\zeta}$ is continuous. Then we claim that $\{\tilde{y}_0, \cdots, \tilde{y}_{n-1}, y_n\}$ is contained in Λ. Obviously y_n belongs to Λ. $d(y_n, f(y_{n-1})) < r$ implies $d(h^{-1}(y_n), y_{n-1}) < \zeta$. If $h^{-1}(y_n) \ne \tilde{y}_{n-1}$, then there is $z \in [y_{n-1}, h^{-1}(y_n)]_{\varepsilon_1,\zeta}$. Then $z \in W^s_{\varepsilon_1}(y_{n-1})$ so $f(z) \in W^s_{\lambda\varepsilon_1}(f(y_{n-1}))$. On the other hand,

$$f(W^u_{\varepsilon_1}(h^{-1}(y_n))) \supset W^u_{\varepsilon_1}(y_n) \ni f(y_{n-1}), f(z).$$

Hence $f(z) \in W^u_{\varepsilon_1}(f(y_{n-1})) \cap W^s_{\varepsilon_1}(f(y_{n-1})) = \{f(y_{n-1})\}$ so

$$h(z) = f(z) = f(y_{n-1}) = h(y_{n-1}).$$

By the injectivity of h, $z = y_{n-1}$. Then $h^{-1}(y_n)$, $\tilde{y}_{n-1} \in W^u_{\varepsilon_1}(y_{n-1})$ by self-coherence of the disk family $\{W^u_{\varepsilon_1}(\cdot)\}$. Moreover $y_n = f(\tilde{y}_{n-1}) = f(h^{-1}(y_n))$. This contradicts the injectivity of $f|W^u_{\varepsilon_1}(y_{n-1})$. Therefore $\tilde{y}_{n-1} = h^{-1}(y_n) \in \Lambda$. By the above argument for a pair $\{y_{n-1}, f(y_{n-2})\}$ instead of $\{y_n, f(y_{n-1})\}$, there exists $\overline{y}_{n-2} = h^{-1}(y_{n-1}) \in \Lambda$ such that $\overline{y}_{n-2} \in W^u_{\varepsilon_1}(y_{n-2})$ and $d(\overline{y}_{n-2}, y_{n-2}) < \lambda r < r$. Since $\tilde{y}_{n-1} \in W^u_{\lambda\varepsilon_1}(y_{n-1})$ and $d(\tilde{y}_{n-1}, y_{n-1}) < \lambda r$, $d(h^{-1}(y_{n-1}), h^{-1}(\tilde{y}_{n-1})) < \zeta$. Hence $d(\overline{y}_{n-2}, h^{-1}(\tilde{y}_{n-1})) < \zeta$. If $h^{-1}(\tilde{y}_{n-1}) \ne \tilde{y}_{n-2}$, then there is $u \in [\overline{y}_{n-2}, h^{-1}(\tilde{y}_{n-1})]$. Then $u \in W^s_{\varepsilon_1}(\overline{y}_{n-2})$ so $f(u) \in W^s_{\lambda\varepsilon_1}(y_{n-1})$. On the other hand,

$$f(W^u_{\varepsilon_1}(h^{-1}(\tilde{y}_{n-1}))) \supset W^u_{\varepsilon_1}(\tilde{y}_{n-1}) \ni y_{n-1}, f(u).$$

Hence $f(u) = W^u_{\varepsilon_1}(y_{n-1}) \cap W^s_{\varepsilon_1}(y_{n-1}) = \{y_{n-1}\}$ so $u = h^{-1}(y_{n-1})$ because u, y_{n-1} belong to Λ. Thus $u = \overline{y}_{n-2}$ so $h^{-1}(\tilde{y}_{n-1}), \tilde{y}_{n-2} \in W^u_{\varepsilon_1}(\overline{y}_{n-2})$, and $\tilde{y}_{n-1} = f(h^{-1}(\tilde{y}_{n-1})) = f(\tilde{y}_{n-2})$. Hence $\tilde{y}_{n-2} = h^{-1}(\tilde{y}_{n-1})$ by injectivity of $f|W^u_{\varepsilon_1}(\overline{y}_{n-2})$. By the similar argument, we obtain that $\tilde{y}_i$ belongs to Λ for $0 \le i \le n-3$. The case of an arbitrary finite pseudo-orbit proceeds as above. Finally, if $\underline{x}$ is an infinite pseudo-orbit, we can find y_n in Λ which β-shadows a finite segment $\{x_{-n}, \cdots, x_0, \cdots, x_n\}$ of $\underline{x}$ and $d(f^n(y_n), x_0) < \beta$. Since Λ is compact, $\{f^n(y_n)\}$ has a limit point y in Λ which β-shadows $\underline{x}$. $\square$

Proposition 2.5. *Let ε be a constant of expansivity of f on Λ as in Proposition* 2.4 *and let γ be a positive constant less than $\varepsilon/2$. Then*

(a) *A bi-infinite pseudo-orbit $\underline{x}$ is γ-shadowed by at most one point y in Λ.*

(b) *There are a constant α and a neighborhood U of Λ such that every α-pseudo-orbit $\underline{x}$ in U is γ-shadowed by a point y in Λ. If, moreover, $\underline{x}$ is bi-infinite, then y is unique.*

Proof. (a) By Proposition 2.4 there is a point y in Λ which γ-shadows $\underline{x}$. Let $\underline{z}$ be a bi-infinite orbit which shadows $\underline{x}$. Then $d(z_n, h^n(y)) \leq 2\gamma < \varepsilon$ for all $n \in Z$, where $h = f|\Lambda$ is a homeomorphism. Hence $z_n = h^n(y)$ for all $n \in Z$ so y and z_0 coincide.

(b) Let α_1 correspond to the choice of $\beta = \gamma/2$ in Proposition 2.4. Choose a neighborhood U of Λ and a constant α such that every α-pseudo-orbit $\underline{x}$ in U is approximated to within $\gamma/2$ by an α_1-pseudo-orbit $\underline{x}'$ in Λ. The α_1-pseudo-orbit $\underline{x}'$ is $\gamma/2$-shadowed by a point y of Λ, which also γ-shadows $\underline{x}$. It follows from (a) uniqueness of y for a bi-infinite α-pseudo-orbit $\underline{x}$. □

Definition. Let $f \in \mathrm{End}^r(M)$, $r \geq 1$, and Λ be a compact f-invariant set. We say that Λ is *locally maximal* if there is a neighborhood U of Λ in M such that $\bigcap_{m\in Z} f^m(U) = \Lambda$.

Pseudo-orbits and Shadowing Lemma are very useful tools to prove that a prehyperbolic set with a local product structure is locally maximal, as follows.

Proposition 2.6. *Let Λ be a prehyperbolic set for $f \in \mathrm{End}^r(M)$, $r \geq 1$. Suppose that Λ has a local product structure and the stable index $i > 0$. Then there are neighborhoods U of Λ in M and V of f in $\mathrm{End}^r(M)$ such that*:

(1) $\Lambda = \bigcap_{n\in Z} f^n(U)$;

(2) *The set $\Phi(g)\Lambda$ given by Theorem* 2.2 *for g in V, is equal to $\bigcap_{n\in Z} g^n(U)$.*

Proof. (1) Let γ be a positive constant less than $\varepsilon/2$, where ε is a constant of expansivity of f on Λ. Let α and U be a constant and a neighborhood of Λ in M corresponding to γ in Proposition 2.5. If z belongs to $\bigcap_{n\in Z} f^n(U)$, then there exists a bi-infinite f-orbit $\{z_i\}$ such that $z_0 = z$, $z_i \in U$ for all $i \in Z$. By Proposition 2.5(b) there is a unique point y in Λ, which γ-shadows $\{z_i\}$. Hence $d(h^i(y), z_i) < \gamma < \varepsilon$ for all $i \in Z$, where $h = f|\Lambda$ is a homeomorphism. Expansivity of f on Λ implies $y = z_0$. Therefore $\bigcap_{n\in Z} f^n(U) \subset \Lambda$. It is obvious that $\Lambda \subset \bigcap_{n\in Z} f^n(U)$.

(2) Let U be a neighborhood of Λ as in (1). Let V be a very small neighborhood of f where the map Φ of Theorem 2.2 is defined and, furthermore, for all g in V, $\sup_{x\in M} d(g(x), f(x)) < \alpha' < \alpha$, where α is as in Proposition 2.4. By (1) we shrink V if necessary so that if z belongs to $\bigcap_{n\in Z} g^n(U)$, there exists a bi-infinite g-orbit $\{z_i\}$ which is a bi-infinite α'-pseudo-orbit of f, contained in U, and satisfies $z_0 = z$. By Proposition 2.4, $\{z_i\}$ is γ-shadowed by a unique point x in Λ. That is, $d(h^n(x), z_n) < \gamma$ for all $n \in Z$. We claim that $\Phi(g)(x) = z$. Set $y_n = \Phi(g)(h^n(x))$ for all $n \in Z$, where $h = f|\Lambda$ is a homeomorphism of Λ onto itself. Then by taking V small enough, $\{y_n\}$ is a bi-infinite g-orbit such that

$$d(y_n, h^n(x)) = d(\Phi(g)(h^n(x)), h^n(x)) \leq d(\Phi(g), id) < \gamma \quad \text{for all } n \in Z.$$

Next, consider the diagram

$$\begin{array}{ccc} Z & \xrightarrow[i]{} & U \\ \sigma\big\downarrow & & g\big\downarrow \\ Z & \xrightarrow[i]{} & M \end{array}, \text{ where } \ i(n) = h^n(x) \ \text{ and } \ \sigma(n) = n + 1.$$

Theorem 2.1 guarantees that for V sufficiently small, there is a unique continuous map $j : Z \to M$ such that $j\sigma = gj$ and $d(i, j) < \gamma$. Then it is easy to see that $\{j(n)|n \in Z\}$ is a unique bi-infinite g-orbit such that

(a) $d(h^n(x), j(n)) < \gamma$ for all $n \in Z$.

By uniqueness of bi-infinite g-orbit satisfying (a), if γ is small enough (that is, V is small enough), $\{z_n\} = \{j(n)\}$. Hence $z = z_0 = j(0) = \Phi(g)(x)$. Therefore $\bigcap_{n\in Z} g^n(U) \subset \Phi(g)(\Lambda)$. By the continuity of Φ, we can take a small neighborhood V such that $\Phi(g)(\Lambda)$ is contained in U for all g in V. Then $\Phi(g)(\Lambda) \subset \bigcap_{n\in Z} g^n(U)$. □

Proposition 2.7. *Let Λ be a prehyperbolic set for $f \in \mathrm{End}^r(M)$, $r \geq 1$. Suppose that Λ has a local product structure and the stable index > 0. We have*

$$\widetilde{\widetilde{W^s}}(\Lambda) = \bigcup_{x\in\Lambda} \widetilde{W^s}(x) = \widetilde{W^s}(\Lambda) \quad \textit{and} \quad \widetilde{\widetilde{W^u}}(\Lambda) = \bigcup_{x\in\Lambda} W^u(x) = W^u(\Lambda).$$

Proof. By Proposition 2.5, if we are given a sufficiently small δ, we can find a neighborhood U of Λ and a $\alpha > 0$ such that every α-pseudo-orbit of f in U is δ-shadowed by a point of Λ. For an arbitrary y in $\widetilde{\widetilde{W^s}}(\Lambda)$ there is a positive integer N so large that

$$f^n(y) \in U \quad \text{for all} \quad n \geq N.$$

Now the set $\underline{y} = \{y_i|y_i = f^{i+N}(y), i \geq 0\}$ is a positive orbit of $f^N(y)$ in U, and is therefore δ-shadowed by some x in Λ, that is,

$$d(f^i(x), f^{i+N}(y)) < \delta \ \text{ for all } \ i \geq 0.$$

If δ is small enough so that the local stable set $W^s_\delta(x)$ is defined, $f^N(y)$ must belong to $W^s_\delta(x)$, so $y \in \widetilde{W^s}(h^{-N}(x))$, where $h = f|\Lambda$ is a homeomorphism. Hence

$$\widetilde{\widetilde{W^s}}(\Lambda) \subset \bigcup_{x\in\Lambda} \widetilde{W^s}(x).$$

It is obvious that $\widetilde{\widetilde{W^s}}(\Lambda) \supset \bigcup_{x\in\Lambda} \widetilde{W^s}(x)$.

For z in $\widetilde{\widetilde{W^u}}(\Lambda)$ there is a positive integer L so large that $z_{-n} \in U$ for all $n \geq L$, where $\{z_{-n}\}$ is an infinitely negative orbit of z. Now the set $\underline{a} = \{a_{-i}|a_{-i} = z_{-L-i}, \ i \geq 0\}$ is a negative orbit of z_{-L} in U, and is therefore δ-shadowed by some b in Λ, that is,

$$d(h^{-i}(b), a_{-i}) < \delta \ \text{ for all } \ i \geq 0.$$

If δ is small enough so that the local unstable set $W^u_\delta(b)$ is defined, z_{-L} must belong to $W^u_\delta(b)$, so $z \in W^u(f^L(b))$. Hence $\widetilde{\widetilde{W^u}}(\Lambda) \subset \bigcup_{x\in\Lambda} W^u(x)$. Obviously $\bigcup_{x\in\Lambda} W^u(x) \subset \widetilde{\widetilde{W^u}}(\Lambda)$. □

By the above result we will adopt $\widetilde{W^s}(\Lambda)$ and $W^u(\Lambda)$ for the definition of the no-cycles condition of weak Axiom A endomorphisms. If f is a weak Axiom A endomorphism, $\Omega(f)$ has a unique decomposition $\Omega(f) = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_k$ into disjoint f-invariant compact sets. Here each Λ_i is a prehyperbolic set such that every point in $\mathrm{Per}(f) \cap \Lambda_i$ has same stable index. We may define a preorder $\gg$ on the Λ_i's by $\Lambda_i \gg \Lambda_j$ iff

$$(W^u(\Lambda_i) - \Lambda_i) \cap (\widetilde{W^s}(\Lambda_j) - \Lambda_j) \neq \phi.$$

We say that the preorder has an *r-cycle* if there is a sequence $\Lambda_{i_1} \gg \cdots \gg \Lambda_{i_{r+1}} = \Lambda_{i_1}$. We say that f satisfies the *no-cycles condition* if the preorder has no r-cycles for all $1 \leq r \leq k+1$. Remark that a diffeomorphism satisfying Axiom A has no 1-cycles [12].

In the case of endomorphisms we cannot directly use the abstract theory of filtrations for diffeomorphisms or homeomorphisms in Chapters 2, 3 of [12]. However, using the following arguments we can prove the filtration lemma for a weak Axiom A endomorphism with the no-cycles condition.

Lemma 2.8. *Let $f \in \mathrm{End}^r(M)$, $r \geq 1$, satisfy weak Axiom A. Let Λ_i, Λ_j be prehyperbolic sets in the decomposition of $\Omega(f)$. If $\overline{W^u(\Lambda_i)} \cap \Lambda_j \neq \phi$, $i \neq j$, then $\overline{W^u(\Lambda_i)} \cap (\widetilde{W^s}(\Lambda_j) - \Lambda_j) \neq \phi$.*

Proof. Choose small compact sets U_k which contain the Λ_k's in their interior with the property that $f(U_k) \cap U_m = \phi$ for $k \neq m$. If $W^u(\Lambda_i) \cap \Lambda_j \neq \phi$ then it is obvious that $(W^u(\Lambda_i) - \Lambda_i) \cap (\widetilde{W^s}(\Lambda_j) - \Lambda_j) \neq \phi$. So we suppose that $W^u(\Lambda_i) \cap \Lambda_j = \phi$.

Case 1. there is a point x in $W^u(\Lambda_i) - (\Lambda_i \cup \Lambda_j)$ such that $\{f^n(x) | n \geq 0\}$ has a limit point in Λ_j. Then there is a constant $N > 0$ such that $f^n(x) \in U_j$ for all $n \geq N$. Hence $d(f^n(x), \Lambda_j) \to 0$ as $n \to \infty$. By Proposition 2.7, $x \in \widetilde{W^s}(\Lambda_j) - \Lambda_j$.

Case 2. there is no point such as in Case 1. Then we can choose a sequence of points of distinct orbits x_n in $W^u(\Lambda_i) - \Lambda_i$ converging to Λ_j. It is no loss of generality to choose them in $Int(U_j)$. Now the definition of $W^u(\Lambda_j)$ allows us to find, for each x_n, a point y_n in $W^u(\Lambda_i) - \Lambda_i$, a least positive integer k_n with $f^{k_n}(y_n) = x_n$ and $y_n \notin Int(U_j)$. Let x be a limit point of the sequence $\{y_n\}$ and notice $x \in \overline{W^u(\Lambda_i)}$.

Subcase A. $k_n \to \infty$ as $n \to \infty$. We claim that $f^m(x)$ is in U_j for all $m > 0$. If this claim holds then we find $x \in \widetilde{W^s}(\Lambda_j) - \Lambda_j$. Now we have that x is not in $Int(U_j)$ and $f(x) \in U_j$. If there is a positive integer $m \geq 2$ with $f^m(x)$ in $M - U_j$ and $f^{m-1}(x) \in U_j$, then there is an y_n near x with $k_n > m$ so $f^m(y_n) \in M - U_j$. This is a contradiction to the minimality of k_n.

Subcase B. $\{k_n\}$ is bounded. Then we can suppose that $f^k(y_n) = x_n$ for all $n > 0$, where k is some positive integer. Hence $f^k(x) = \lim_{n\to\infty} f^k(y_n) = \lim_{n\to\infty} x_n$. Therefore $f^k(x) \in \Lambda_j$ so $x \in \widetilde{W^s}(\Lambda_j) - \Lambda_j$. □

Proposition 2.9. *If $f \in \mathrm{End}^r(M)$, $r \geq 1$, satisfies weak Axiom A and the no-cycles condition, then for any family of compact neighborhoods U_i of Λ_i there exists an adapted filtration, i.e. there exists a finite sequence of compact sets $M_0, \cdots, M_l$ such that*

(1) $\phi = M_0 \subset M_1 \subset \cdots \subset M_l = M$;
(2) $f(M_i) \subset Int(M_i)$ *for every* i ;
(3) $\Lambda_i = \bigcap_{n=-\infty}^{+\infty} f^n(M_i - M_{i-1}) \subset U_i$.

Proof. Since f satisfies weak Axiom A then $\Omega(f)$ has a unique decomposition $\Omega(f) = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_k$ into disjoint compact sets. Here each Λ_i is a prehyperbolic set. Remark that $\mathrm{Per}(f) \cap \Lambda_i$ is dense in Λ_i for $0 \le i \le k$. Moreover, all points in $\mathrm{Per}(f) \cap \Lambda_i$ have same stable index. Hence each Λ_i has a local product structure. By the no-cycles condition there is $0 \le j \le k$ such that Λ_j is an attractor, i.e., there is a neighborhood U of Λ_j with $\bigcap_{n\ge 0} f^n(U) = \Lambda_j$. In other words $W^u(\Lambda_j) = \Lambda_j$. Then there exists a compact neighborhood M_1 of Λ_j such that $\bigcap_{m\ge 0} f^m(M_1) = \Lambda_j$, $f(M_1) \subset Int(M_1)$ and $M_1 \cap (\bigcup_{0\le i\le k}^{i\ne j} \Lambda_i) = \phi$ [9], [13]. Since there are no cycles and $\{\Lambda_i\}$ is finite, there exists $0 \le j_1 \le k$, $j_1 \ne j$ such that $\Lambda_{j_1} \gg \Lambda_j$ but $\Lambda_{j_1} \not\gg \Lambda_i$ for all $0 \le i \le k$, $i \ne j$. We claim that

$$(\Lambda_j \cup \overline{W^u(\Lambda_{j_1})}) \cap \left(\bigcup_{i\ne j_1, j} \Lambda_i \right) = \phi.$$

It is obvious that $\Lambda_j \cap (\bigcup_{i\ne j_1,j} \Lambda_i) = \phi$. So we suppose that

$$\overline{W^u(\Lambda_{j_1})} \cap \Lambda_l \ne \phi \ \text{ for some } l \ne j_1, j.$$

By Lemma 2.8, $\overline{W^u(\Lambda_{j_1})} \cap (\widetilde{W^s}(\Lambda_l) - \Lambda_l) \ne \phi$. Let $x \in \overline{W^u(\Lambda_{j_1})} \cap (\widetilde{W^s}(\Lambda_l) - \Lambda_l)$.

By the selection of j_1, $x \notin W^u(\Lambda_{j_1})$. Hence we have only possibility as in Case 2 in the proof of Lemma 2.8. Then $x \in \overline{W^u(\Lambda_{j_1})}$ is a limit point of x_n in $W^u(\Lambda_{j_1}) - \Lambda_{j_1}$. Remark that each x_n has an infinitely negative orbit. So x has an infinitely negative orbit $\{z_m\}$ in $\overline{W^u(\Lambda_{j_1})}$ but not in $W^u(\Lambda_{j_1})$. Then $\alpha(\{z_m\}) \subset \Omega(f)$, where $\alpha(\{z_m\}) = \{y \in M|$ there is a decreasing sequence m_q of negative integers such that $\lim_{q\to+\infty} z_{m_q} = y\}$. It is easy to see that $\alpha(\{z_m\}) \subset \Lambda_p$ for some $p \ne j, j_1$. Hence $x \in (W^u(\Lambda_p) - \Lambda_p) \cap (\widetilde{W^s}(\Lambda_l) - \Lambda_l)$ so $\Lambda_p \gg \Lambda_l$. Now notice that $x \in \overline{W^u(\Lambda_{j_1})} \cap (W^u(\Lambda_p) - \Lambda_p)$ and $\alpha(\{z_m\}) \subset \overline{W^u(\Lambda_{j_1})}$. Therefore $\overline{W^u(\Lambda_{j_1})} \cap \Lambda_p \ne \phi$.

Repeating the above argument using Lemma 2.8 we construct a sequence $\cdots \gg \Lambda_r \gg \cdots \gg \Lambda_p \gg \Lambda_l$. Since $\{\Lambda_i\}$ is finite, we have a cycle. This contradicts to the no-cycles condition. We can take a compact neighborhood Q_2 of $\Lambda_j \cup \overline{W^u(\Lambda_{j_1})}$ such that

$$\Lambda_j \cup \overline{W^u(\Lambda_{j_1})} \subset \bigcap_{m\ge 0} f^m(Q_2) \quad \text{and} \quad Q_2 \cap \left(\bigcup_{i\ne j, j_1} \Lambda_i \right) = \phi.$$

We shall show that $\Lambda_j \cup W^u(\Lambda_{j_1}) = \bigcap_{m\ge 0} f^m(Q_2)$. Let $x \in \bigcap_{m\ge 0} f^m(Q_2)$. Then there is an infinitely negative orbit $\{x_{-n}|n \ge 0\} \subset Q_2$ with $x_0 = x$. Hence $\alpha(\{x_{-n}\}) \subset Q_2$. Here we shall prove that $x \in \Lambda_j \cup W^u(\Lambda_{j_1})$. Since $\alpha(\{x_{-n}\}) \subset Q_2$, $\alpha(\{x_{-n}\}) \subset \Lambda_j \cup \Lambda_{j_1}$. Then it is easy to see that $\alpha(\{x_{-n}\}) \subset \Lambda_j$ or $\alpha(\{x_{-n}\}) \subset \Lambda_{j_1}$. It follows from Proposition 2.7 that $x \in \Lambda_j \cup W^u(\Lambda_{j_1})$. Then there exists a compact neighborhood M_2 of $\Lambda_j \cup W^u(\Lambda_{j_1})$ such that $\bigcap_{m\ge 0} f^m(M_2) = \Lambda_j \cup W^u(\Lambda_{j_1})$, $M_1 \subset M_2$, $f(M_2) \subset Int(M_2)$ and $M_2 \cap \bigcap_{i\ne j_1,j} \Lambda_i = \phi$.

Proceeding inductively we produce a nest sequence $\{M_i\}$ satisfying (1), (2). For simplicity reindex so that

$$\Lambda_{n(j)} \subset M_{j+1} - M_j \quad \text{for} \quad 0 \leq j < k.$$

Finally we check that $\Lambda_{n(j)} = \bigcap_{m \in Z} f^m(M_{j+1} - M_j)$. If $x \in \bigcap_{m \in Z} f^m(M_{j+1} - M_j)$, there exists an infinitely negative orbit $\{x_{-n} | n \geq 0\}$ such that $x_0 = x$, $x_{-n} \in M_{j+1} - M_j$ for all $n \geq 0$. Then $\alpha(\{x_{-n}\}) \subset Cl(M_{j+1} - M_j)$. Hence $\alpha(\{x_{-n}\}) \subset \Lambda_{n(j)}$ so $x \in W^u(\Lambda_{n(j)})$. Similarly $\omega(x) \subset \Lambda_{n(j)}$ so $x \in \widetilde{W^s}(\Lambda_{n(j)})$. Therefore $x \in W^u(\Lambda_{n(j)}) \cap \widetilde{W^s}(\Lambda_{n(j)})$. Since there are no 1-cycles, $x \in \Lambda_{n(j)}$. This proves (3). □

Proposition 2.10. *Let $f \in \mathrm{End}^r(M)$, $r \geq 1$, satisfy weak Axiom A and the no-cycles condition. Let $(M_0, \cdots, M_k)$ be a filtration adapted to f. Then there exist a neighborhood $\mathcal{N}$ of f in $\mathrm{End}^r(M)$ and a family of compact neighborhoods $\{U_i\}$ such that for each g in $\mathcal{N}$,*

(1) *$\bigcap_{m \in Z} g^m(U_i)$ is locally maximal with respect to U_i ;*
(2) *$g(M_i) \subset Int(M_i)$;*
(3) *$\Omega_i(g) = \Omega(g) \cap (M_i - M_{i-1}) = \bigcap_{m \in Z} g^m(U_i)$.*

Proof. Let $\Omega(f) = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_{k-1}$ be a unique decomposition as in Proposition 2.9 such that $\Lambda_i \subset M_{i+1} - M_i$ for all $0 \leq i < k$. By Proposition 2.6, for each $0 \leq i < k$ there exist neighborhoods U_i of Λ_i in M and $\mathcal{U}_i$ of f in $\mathrm{End}^r(M)$ such that every g in $\mathcal{U}_i$ satisfies (1), (2). Let $\mathcal{U} \subset \bigcap_{i=0}^{k-1} \mathcal{U}_i$ be small enough so that every g in $\mathcal{U}$ satisfies (1), (2) for each $0 \leq i < k$. It remains only to show (3). First notice that each Λ_i is the maximal f-invariant subset in $M_{i+1} - M_i$. On the other hand for each g in $\mathcal{U}$ $\Phi(g)(\Lambda_i)$ is g-invariant subset in $M_{i+1} - M_i$ for each $0 \leq i < k$. It is easy to see that $\Phi(g)(\Lambda_i) \subset \Omega_i(g) = \Omega(g) \cap (M_{i+1} - M_i)$. Remark that

$$\Omega_i(f) = \Lambda_i = \bigcap_{m \in Z} f^m(U_i) = \bigcap_{m \in Z} f^m(M_{i+1} - M_i).$$

So we can choose a positive integer N such that

$$\bigcap_{-N < m < N} f^m(M_{i+1} - M_i) \subset Int(U_i) \quad \text{for each } i = 0, 1, \cdots, k-1.$$

Also for g sufficiently C^r close to f, we have

$$\bigcap_{-N < m < N} g^m(M_{i+1} - M_i) \subset Int(U_i),$$

so

$$\bigcap_{m \in Z} g^m(M_{i+1} - M_i) = \bigcap_{m \in Z} g^m(U_i) = \Phi(g)(\Lambda_i).$$

That is, $\Phi(g)(\Lambda_i)$ is the maximal g-invariant subset in $M_{i+1} - M_i$ for each $i = 0, 1, \cdots, k-1$. Since $\Phi(g)(\Lambda_i) \subset \Omega_i(g)$ we only need to prove that $\Omega_i(g)$ is g-invariant. Suppose that $g(\Omega_i(g)) \neq \Omega_i(g)$. Then there is $x \in \Omega_i(g)$ such that $g^{-1}(x) \cap \Omega(g) = \phi$. Since $x \in \Omega(g)$, there exists a sequence of g-orbits $\{\{z_n^k\} | k = 1, 2, \cdots\}$ such that:

(a) $\{z_0^k, z_{-1}^k, \cdots, z_{n(k)}^k\}$ is a negative orbit of z_0^k for each $k \in N$;
(b) $\lim_{k \to +\infty} n(k) = -\infty$;
(c) $\lim_{k \to +\infty} z_0^k = \lim_{k \to +\infty} z_{n(k)}^k = x$.

From the above sequence $\{\{z_n^k\}|k = 1, 2, \cdots\}$ we can get an infinitely negative orbit of x, $\{x_n\}$. Since $g^n(x) \in \Omega_i(g)$ for $n \geq 0$, $g^n(x) \in M_{i+1} - M_i$ for all $n \geq 0$. Hence there is a negative integer q such that $x_q \notin M_{i+1}$. Therefore there exists a positive integer p such that $z_q^p \notin M_i$, $z_{n(p)}^p \in M_i$ and $n(p) < q$. This contradicts (2). □

3. Proof of Theorem

Let Λ be a prehyperbolic set for $f \in \mathrm{End}^r(M)$, $r \geq 1$. By Theorem 2.1 there are numbers $\alpha > 0$, $K > 0$, $k > 0$, a neighborhood U of Λ in M and a neighborhood V of f in $\mathrm{End}^r(M)$. For $X = \Lambda$, $h = f|\Lambda$, $i =$ inclusion map of Λ into M in Theorem 2.1, if $g \in V$ and $d(f, g) < \alpha$ then there is a unique continuous map $\Phi(g) : \Lambda \to M$ such that $\Phi(g) \circ f = g \circ \Phi(g)$ and $d(i, \Phi(g)) < Kd(f, g)$ and $d(i, \Phi(g)) \leq k$.

Claim. There exists a C^r neighborhood $\mathcal{U}$ of f in $\mathrm{End}^r(M)$ such that $\Phi(g)(\Lambda)$ is P-hyperbolic for every g in $\mathcal{U}$.

Note that the above Claim is property (2) in Theorem 2.2.

Proof. If the stable index of Λ is zero then $f|\Lambda$ is a (quasi-) expanding map. Then it is easy to see that there exist neighborhoods U_0 of Λ and $\mathcal{U}^0$ of f such that if g belongs to $\mathcal{U}^0$ then there is a homeomorphism $\Phi_0(g)$ from Λ onto its image satisfying $\Phi_0(g) \circ f = g \circ \Phi_0(g)$ [9]. Moreover $g|\Phi_0(g)(\Lambda)$ is (quasi-) expanding for each g in $\mathcal{U}^0$. Suppose that the stable index of $\Lambda = i \neq 0$, i.e. $0 < i \leq \dim M$. Then there exist a continuous splitting $TM|\Lambda = E^s \oplus E^u$, and a Riemannian norm $|\cdot|$ on TM, and constants $K > 0$, $0 < \lambda < 1$ satisfying:

(a) $(Tf)E^s \subset E^s \quad , \quad (Tf)E^u = E^u$;
(b) $|(Tf)^n v| \leq K\lambda^n |v| \quad$ for $v \in E_x^s$, $x \in \Lambda$, $n > 0$,

$$|(Tf)^n v| \geq K\lambda^{-n}|v| \quad \text{for} \quad v \in E_x^u,\ x \in \Lambda,\ n > 0;$$

(c) $f|\Lambda$ is a homeomorphism from Λ to itself.

Moreover we can take an adapted Riemannian norm on TM such that the above $K = 1$. First we continuously extend the bundle $E^s \oplus E^u$ over Λ to bundles $\widetilde{E}^s$, $\widetilde{E}^u$ over an open neighborhood W. If $x \in W \cap f^{-1}(W)$ we can write $Tf : \widetilde{E}_x^s \oplus \widetilde{E}_x^u \to \widetilde{E}_{f(x)}^s \oplus \widetilde{E}_{f(x)}^u$ as a block matrix

$$\begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix}.$$

Let $\delta > 0$ be a sufficiently small constant such that $\lambda + 2\delta < 1$. Let U_1 be a neighborhood of Λ such that

$$U_1 \subset W, ||A_x|| < \lambda + \frac{\delta}{2}, ||B_x|| < \frac{\delta}{2}, ||C_x|| < \frac{\delta}{2}, ||D_x^{-1}|| < \lambda + \frac{\delta}{2} \ \text{ for all } x \in U_1.$$

Moreover let U_2 be a neighborhood of Λ with compact closure such that $\overline{U_2} \cup \overline{f(U_2)} \subset U_1$. If g is C^r close to f, the neighborhood U_1 will contain $g(U_2)$ and the linear map $Tg_x : \widetilde{E}_x^s \oplus \widetilde{E}_x^u \to \widetilde{E}_{g(x)}^s \oplus \widetilde{E}_{g(x)}^u$, $x \in U_2$, has the matrix $\begin{pmatrix} \overline{A}_x & \overline{B}_x \\ \overline{C}_x & \overline{D}_x \end{pmatrix}$ with

$$||\overline{A}_x|| < \lambda + \delta,\ ||\overline{B}_x|| < \delta,\ ||\overline{C}_x|| < \delta,\ ||\overline{D}_x^{-1}|| < \lambda + \delta.$$

Take a neighborhood $\mathcal{U} \subset V$ of f in $\mathrm{End}^r(M)$ such that

$$\Phi_i(g)(\Lambda) \subset U_2 \quad \text{for all} \quad g \in \mathcal{U}.$$

For $x \in U_1$, $\varepsilon > 0$, let

$$S^s_\varepsilon(x) = \{(v_1, v_2) \in \widetilde{E}^s_x \oplus \widetilde{E}^u_x | \quad |v_2| \leq \varepsilon |v_1|\},$$
$$S^u_\varepsilon(x) = \{(v_1, v_2) \in \widetilde{E}^s_x \oplus \widetilde{E}^u_x | \quad |v_1| \leq \varepsilon |v_2|\}$$

where $|\cdot|$ is the adapted Riemannian norm in T_xM. Then if necessary shrinking $\mathcal{U}$, U_2, there exists a constant $0 < \mu < 1$ such that each $g \in \mathcal{U}$ satisfies: for every $x \in U_2$

$$||(Tg)|\widetilde{E}^s_x|| \leq \lambda + \frac{3}{2}\delta < \mu < 1, \quad |(Tg)v| \geq \mu^{-1}|v| \quad \text{for} \;\; v \in \widetilde{E}^u_x.$$

If $\varepsilon > 0$ is small enough and if necessary shrinking $\mathcal{U}$, U_2 we can take $\mu < \bar{\mu} < 1$ such that for all $x \in U_2$, all $g \in \mathcal{U}$, $v \in S^u_\varepsilon(x)$, $w \in S^s_\varepsilon(x)$ satisfy:

(1) $|(Tg)w| \leq \bar{\mu}|w|$

(2) $|(Tg)v| \geq \bar{\mu}^{-1}|v|$

(3) $(Tg_x)(S^u_\varepsilon(x)) \subset S^u_{\bar{\mu}\varepsilon}(g(x))$

(4) $(Tg_x)^{-1}(S^s_\varepsilon(g(x))) \subset S^s_{\bar{\mu}\varepsilon}(x)$.

For each $g \in \mathcal{U}$ let $\widetilde{\Lambda}_g = \widetilde{\Phi_i(g)(\Lambda)}$ be the set of all bi-infinite orbits contained in $\Phi_i(g)(\Lambda)$. We define the following relation $\simeq$ on $\widetilde{\Lambda}_g$: for $\underline{x}$, $\underline{y}$ in $\widetilde{\Lambda}_g$, $\underline{x} \simeq \underline{y}$ if $\widetilde{f}^l(\underline{x}) = \underline{y}$ or $\widetilde{f}^l(\underline{y}) = \underline{x}$ for some integer $l \geq 0$.

Let $\langle \underline{x} \rangle$ be an equivalent class containing $\underline{x}$. Let $\hat{\Lambda}_g$ be the set of all equivalent classes. For each $\langle \underline{x} \rangle$ in $\hat{\Lambda}_g$, set

$$E^s_{x_l}(\langle \underline{x} \rangle, n) = (Tg)^{-n}(\widetilde{E}^s_{x_{n+l}}), \quad E^u_{x_l}(\langle \underline{x} \rangle, n) = (Tg)^n(\widetilde{E}^u_{x_{-n+l}})$$

for every $n \in Z^+$, where (x_l) is a representative of $\langle \underline{x} \rangle$.

By Tychonoff's theorem there exist an increasing sequence $\{n_j | j \in Z^+\} \subset Z^+$ and a sequence $\{\overline{E}^s_{x_l}(\langle \underline{x} \rangle) | l \in Z\}$ such that

$$\overline{E}^s_{x_l}(\langle \underline{x} \rangle) = \lim_{j \to \infty} E^s_{x_l}(\langle \underline{x} \rangle, n_j) \text{ and } \overline{E}^s_{x_l}(\langle \underline{x} \rangle) \subset T_{x_l}M$$

for every $l \in Z$.

Observe that for each $l \in Z$

$$(Tg)\overline{E}^s_{x_l}(\langle \underline{x} \rangle) \subset \lim_{j \to +\infty} (Tg)^{-n_j+1} \widetilde{E}^s_{x_{n_j+l}} \quad \text{and}$$
$$(Tg)^{-n_j+1} \widetilde{E}^s_{x_{n_j+l}} \subset S^s_\varepsilon(x_{l+1}).$$

Hence $(Tg)\overline{E}^s_{x_l}(\langle \underline{x} \rangle) \subset S^s_\varepsilon(x_{l+1})$ and by (1), (4)

$$|(Tg)v| \leq \bar{\mu}|v| \quad \text{for all } v \in \overline{E}^s_{x_l}(\langle \underline{x} \rangle), \qquad l \in Z.$$

By the iteration of the same argument

(5) $|(Tg)^n v| \leq \bar{\mu}^n |v| \quad \text{for all } v \in \overline{E}^s_{x_l}(\langle \underline{x} \rangle), \;\; l \in Z, \quad n \in Z^+.$

By the similar argument for $\overline{E}^u_{x_l}(\langle \underline{x} \rangle, n)$ there exists a sequence $\{\overline{E}^u_{x_l}(\langle \underline{x} \rangle) | \ l \in Z\}$ such that $\overline{E}^u_{x_l}(\langle \underline{x} \rangle)$ is a subspace of $T_{x_l}M$ for every $l \in Z$ and

$$(6) \qquad |(Tg)^n w| \geq \bar{\mu}^{-n} |w| \quad \text{for all} \quad w \in \overline{E}^u_{x_l}(\langle \underline{x} \rangle), \qquad l \in Z, \ n \in Z^+.$$

It is easy to see that $\overline{E}^s_{x_l}(\langle \underline{x} \rangle) \oplus \overline{E}^u_{x_l}(\langle \underline{x} \rangle) = T_{x_l}M$ and $\dim \overline{E}^s_{x_l}(\langle \underline{x} \rangle) = i =$ the stable index of Λ for f for all $l \in Z$. Then it is obvious that

$$(Tg_{x_l})\overline{E}^s_{x_l}(\langle \underline{x} \rangle) \subset \overline{E}^s_{x_{l+1}}(\langle \underline{x} \rangle) \quad \text{and}$$

$$(Tg_{x_l})\overline{E}^u_{x_l}(\langle \underline{x} \rangle) = \overline{E}^u_{x_{l+1}}(\langle \underline{x} \rangle) \quad \text{for all} \quad l \in Z.$$

Hence $\Phi(g)(\Lambda)$ is P-hyperbolic for g in $\mathcal{U}$. □

Proposition 3.1. *If $f \in \mathrm{End}^r(M)$, $r \geq 1$, satisfies weak Axiom A and the no-cycles condition then there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{End}^r(M)$ such that for every g in $\mathcal{U}$ there is a homeomorphism $H_g : \widetilde{\Omega(f)} \to \widetilde{\Phi(g)}(\Omega(f)) = \widetilde{\Omega(g)}$ such that $H_g \tilde{f} = \tilde{g} H_g$ on $\widetilde{\Omega(f)}$, where $\widetilde{\Phi(g)}(\Omega(f))$ is the set of bi-infinite orbits of g contained in $\Phi(g)(\Omega(f))$.*

Proof. Since f satisfies weak Axiom A, $\Omega(f)$ has a decomposition $\Lambda_0(f) \cup \cdots \cup \Lambda_{\dim M}(f)$ into disjoint prehyperbolic sets. Here each $\Lambda_i(f)$ is a prehyperbolic set such that the periodic points in $\Lambda_i(f)$ are dense in $\Lambda_i(f)$ and have the stable index i. Then there exist a continuous splitting $TM|\Omega(f) = E^s \oplus E^u$, and a Riemannian norm $| \ |$ on TM, and constants $K > 0$, $0 < \lambda < 1$ satisfying:

(a) $(Tf)E^s \subset E^s$, $\quad (Tf)E^u = E^u$.

(b) $|(Tf)^n v| \leq K\lambda^n |v| \quad$ for $v \in E^s_x$, $x \in \Omega(f)$, $n > 0$.

(c) $|(Tf)^n v| \geq K\lambda^{-n} |v| \quad$ for $v \in E^u_x$, $x \in \Omega(f)$, $n > 0$.

(d) if $x_1 \neq x_2 \in \Omega(f)$ and $f(x_1) = f(x_2) = y$, then $E^s_y = \{0\}$.

We can take an adapted Riemannian metric $| \ |$ on TM such that $K = 1$. We take a family of neighborhoods $\{W_i\}$ such that W_i is a neighborhood of $\Lambda_i(f)$ for every $0 \leq i \leq \dim M$ and $W_i \cap W_j = \phi$ if $i \neq j$. By Claim for each $\Lambda_i(f)$ there is a neighborhood $\mathcal{U}_i$ of f in $\mathrm{End}^r(M)$ such that $\Phi_i(g)(\Lambda_i(f))$ is P-hyperbolic for g in $\mathcal{U}_i$. Let $\mathcal{U}^0 = \bigcap_{i=0}^{\dim M} \mathcal{U}_i$. Let $\mathcal{U}^1 \subset \mathcal{U}^0$ be such that if $g \in \mathcal{U}^1$ then $Kd(f,g) < \min\{\alpha_i | 0 \leq i \leq \dim M\} = \tilde{\alpha}$, where each α_i is given by Theorem 2.1 for $\Lambda_i(f)$. Since $\Phi_i(g)(\Lambda_i(f))$ is P-hyperbolic for g in $\mathcal{U}^1$, the periodic points of g in $\Phi_i(g)(\Lambda_i(f))$ are prehyperbolic and have the stable index i. Hence it follows from Proposition 2.10 that $P_i(g) \subset \Phi_i(g)(\Lambda_i(f))$ and $Cl[P_i(g)] = \Phi_i(g)(\Lambda_i(f))$ for every $0 \leq i \leq \dim M$, where $P_i(g)$ is the set of periodic points of g with stable index i. For g in $\mathcal{U}^1$ define a map H_i from $\widetilde{\Lambda_i(f)}$ into $\widetilde{\Phi_i(g)}(\Lambda_i(f))$ as follows:

$$H_i(\underline{x}) = (\Phi_i(g)x_n) \quad \text{for } \underline{x} = (x_n) \in \widetilde{\Lambda_i(f)}.$$

Then $H_i(\widetilde{\Lambda_i(f)}) \subset \widetilde{\Phi_i(g)}(\Lambda_i(f))$ and $\tilde{g}H_i = H_i\tilde{f}$ on $\widetilde{\Lambda_i(f)}$, where $\widetilde{\Phi_i(g)}(\Lambda_i(f))$ is the set of bi-infinite orbits of g contained in $\Phi_i(g)(\Lambda_i(f))$. Moreover it is obvious that H_i is continuous and near the inclusion map I_i, that is,

$$\tilde{d}(H_i, I_i) = \sup\{d(H_i(\underline{x}), \underline{x}) | \underline{x} \in \widetilde{\Lambda_i(f)}\} < \tilde{\alpha}.$$

Using the expansiveness of f on $\Lambda_i(f)$, we obtain that H_i is injective provided the C^0 distance of H_i from the inclusion map $I_i : \widetilde{\Lambda_i(f)} \to M^Z$ is less than $\frac{1}{2}\varepsilon_i$, where ε_i is a constant of expansiveness on $\Lambda_i(f)$. Thus, H_i is a homeomorphism from $\widetilde{\Lambda_i(f)}$ into $\widetilde{\Phi_i(g)}(\Lambda_i(f))$. Furthermore it is easy to see that $\mathcal{U}^1 \subset \mathcal{P}F^r(M)$ [5]. Here $\mathcal{P}F^r(M)$ denotes the interior of the set of $f \in \mathrm{End}^r(M)$ such that every periodic point of f is prehyperbolic. Therefore $\Phi_i(g)|P_i(f) : P_i(f) \to P_i(g) \subset \Phi_i(g)(\Lambda_i(f))$ is bijective for every g in $\mathcal{U}^1$. $\Phi_i(g)(\Lambda_i(f))$ is a compact g-invariant P-hyperbolic set for g such that the periodic points of g with stable index i are dense in $\Phi_i(g)(\Lambda_i(f))$. By Theorem 2.2, there exists an ω-invariant set $J_i(f)$ for f such that there is a homeomorphism $\hat{H}_i$ from $\widetilde{\Phi_i(g)}(\Lambda_i(f))$ onto $\widetilde{J_i(f)}$ satisfying $\hat{H}_i\tilde{g} = \tilde{f}\hat{H}_i$ on $\widetilde{\Phi_i(g)}(\Lambda_i(f))$. Let $\widetilde{Cl}[\Phi_i(g)(P_i(f))]$ be the set of all bi-infinite orbits of g contained in $Cl[\Phi_i(g)(P_i(f))]$. Since

$$Cl[\widetilde{\Phi_i(g)}(P_i(f))] = \widetilde{Cl}[\Phi_i(g)(P_i(f))] = \widetilde{\Phi_i(g)}(\Lambda_i(f)),$$

$$\hat{H}_i(\widetilde{\Phi_i(g)}(\Lambda_i(f))) = Cl[\hat{H}_i(\widetilde{\Phi_i(g)}(P_i(f)))] = Cl[\widetilde{P_i(f)}].$$

Hence $J_i(f) = p_0\hat{H}_i(\widetilde{\Phi_i(g)}(\Lambda_i(f))) = Cl[P_i(f)] = \Lambda_i(f)$,
where $p_0 : M^Z \to M$ denotes the 0-th projection. Therefore $\hat{H}_iH_i$ is a homeomorphism from $\widetilde{\Lambda_i(f)}$ to itself with $\hat{H}_iH_i\tilde{f} = \tilde{f}\hat{H}_iH_i$ on $\widetilde{\Lambda_i(f)}$. For g sufficiently C^r near f, $\hat{H}_iH_i$ is the identity map. Hence H_i is onto so H_i is a homeomorphism from $\widetilde{\Lambda_i(f)}$ onto $\widetilde{\Phi_i(g)}(\Lambda_i(f))$. For each $0 \le i \le \dim M$ we take a neighborhood $\widetilde{\mathcal{U}}_i \subset \mathcal{U}^0$ of f in $\mathrm{End}^r(M)$ such that for every g in $\widetilde{\mathcal{U}}_i$ there exists a homeomorphism $H_i : \widetilde{\Lambda_i(f)} \to \widetilde{\Phi_i(g)}(\Lambda_i(f))$ with $H_i\tilde{f} = \tilde{g}H_i$. Let $\mathcal{U} = \bigcap_{i=0}^{\dim M} \widetilde{\mathcal{U}}_i$ and $H_g : \widetilde{\Omega(f)} \to \widetilde{\Omega(g)} = \widetilde{\Lambda_0(g)} \cup \widetilde{\Lambda_1(g)} \cup \cdots \cup \widetilde{\Lambda_{\dim M}(g)}$ a map defined by

$$H_g|\widetilde{\Lambda_i(f)} = H_i : \widetilde{\Lambda_i(f)} \to \widetilde{\Lambda_i(g)} \quad \text{for every} \quad 0 \le i \le \dim M,$$

where $\Lambda_i(g) = \Phi_i(g)(\Lambda_i(f))$. Then for each g in $\mathcal{U}$ there exists a homeomorphism $H_g : \widetilde{\Omega(f)} \to \widetilde{\Omega(g)} = \widetilde{\Phi(g)}(\Omega(f))$ with $H_g\tilde{f} = \tilde{g}H_g$ on $\widetilde{\Omega(f)}$, where $\Phi(g) : \Omega(f) \to \bigcup_{i=0}^{\dim M} \Phi_i(g)(\Lambda_i(f))$ is a map with $\Phi(g)|\Lambda_i(f) = \Phi_i(g)$ for each $0 \le i \le \dim M$. □

4. Concluding remarks

In this section we state the relation between infinitesimal stability and Ω-inverse limit stability. By the result of [5] we now know that if $f \in \mathrm{End}^r(M)$, $r \ge 1$, belongs to the interior of the set of C^r infinitesimally stable endomorphisms of M then f is Ω-stable. Combining the Theorem with the result of [4] we also obtain the following result.

Corollary. *If $f \in \mathrm{End}^r(M)$, $r \ge 1$, is infinitesimally stable then f is Ω-inverse limit stable.*

As a conclusion, it is not known whether an endomorphism f on the boundary of the set of infinitesimally stable endomorphisms is Ω-stable. However an endomorphism f on the boundary of the set of infinitesimally stable endomorphisms is Ω-inverse limit stable.

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